

# Dissipative hydrodynamics for relativistic multi-component systems

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Second-order dissipative hydrodynamic equations for each component of a multi-component system are derived using the entropy principle. The shear viscosity of the whole system, appearing in the equation summed-up over all components, is related to the partial shear pressures and cannot be considered as an external parameter. We demonstrate that it is essential to solve hydrodynamic equations for each component, instead of treating a mixture as an effective one-component system with a free parameter  $\eta/s$ . Thus, extractions of the  $\eta/s$  value of the QGP at RHIC and LHC have to be reexamined.

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The deconfined state of QCD matter produced at the early stage of ultrarelativistic heavy-ion collisions at RHIC and LHC is a multi-component system with quark and gluon degrees of freedom. Large values of elliptic flow coefficient  $v_2$ , observed in experiments of ultrarelativistic heavy-ion collisions at RHIC [1, 2] and LHC [3], indicate that the produced quark-gluon plasma (QGP) is a nearly perfect fluid. This has motivated rapid developments on relativistic dissipative hydrodynamic formalisms [4–8]. The value of the shear viscosity to the entropy density ratio  $\eta/s$  for the QGP at RHIC and LHC was extracted from comparisons of hydrodynamic [9] as well as kinetic transport [10] calculations with experimental data. All these hydrodynamic formalisms are based on the assumption that the quark-gluon mixture can be regarded as an effective one-component system, where  $\eta/s$  is an external parameter characterizing the dissipation in the system. One may ask the question whether this assumption really holds in general, which is of major interest for investigation of the QGP properties.

In this Letter we demonstrate that a standard one-component hydrodynamic description with a single shear viscosity coefficient in general cannot be applied to a multi-component system. We will explain this statement by deriving second-order dissipative hydrodynamic equations for a multi-component system from the entropy principle. Our approach differs from the one reported in Ref. [8], since we introduce separate evolution equations and transport coefficients for each component of the mixture. We then show that by summing-up equations for all components one can obtain an equation for the system as a whole, which has a relaxation-type form characteristic for all the present hydrodynamic formalisms, but the effective shear viscosity for the mixture is now related to the partial shear pressures of its components and thus cannot be considered as an externally specified parameter. It is essential to solve the hydrodynamic equations for each component. We will confirm our findings by comparing the solutions of the hydrodynamic equations

with those from kinetic transport calculations.

We consider a mixture of  $N$  particle species. Neglecting bulk pressure and heat flow we construct the total entropy current as [11]

$$s^\mu = \sum_{i=1}^N s_i^\mu = s_{eq} u^\mu - \sum_{i=1}^N \frac{\beta_i}{2T_i} \pi_{i,\alpha\beta} \pi_i^{\alpha\beta} u^\mu, \quad (1)$$

where  $s_{eq}$  is the total entropy density in local equilibrium and  $u^\mu$  is the hydrodynamic velocity.  $T_i$  and  $e_i$  are the temperature and local energy density of the particle species  $i$ .  $\beta_i = (9/4e_i)$  and  $\pi_i^{\mu\nu} = T_i^{\mu\nu} - T_{i,eq}^{\mu\nu}$  is the shear stress tensor, which is the difference between the energy-momentum tensor  $T_i^{\mu\nu}$  and the equilibrium one. Equation (1) is the generalization of the entropy current for a one-component system ( $N = 1$ ).

The total entropy production is then

$$\partial_\mu s^\mu = \sum_i \pi_{i,\alpha\beta} \left[ \frac{\sigma^{\alpha\beta}}{T_i} - \pi_i^{\alpha\beta} \partial_\mu \left( \frac{\beta_i}{2T_i} u^\mu \right) - \frac{\beta_i}{T_i} u^\mu \partial_\mu \pi_i^{\alpha\beta} \right], \quad (2)$$

where

$$\sigma^{\mu\nu} = \nabla^{(\mu} u^{\nu)} = \left( \frac{1}{2} (\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\alpha^\nu \Delta_\beta^\mu) - \frac{1}{3} \Delta_{\alpha\beta} \Delta^{\mu\nu} \right) \nabla^\alpha u^\alpha$$

and  $\Delta_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta$  with the metric  $g_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$ . We have used conservation of the partial particle flows and total energy-momentum tensor,  $\partial_\mu N_i^\mu = 0$  and  $\partial_\mu T^{\mu\nu} = 0$ , to obtain Eq. (2).

According to the second law of thermodynamics the entropy production is non-negative. A simple way to fulfill this is to make the terms in the square bracket in Eq. (2) to be proportional to  $\pi_i^{\alpha\beta}$ ,  $[\dots] = \pi_i^{\alpha\beta} / (2\eta_i T_i)$ . Then the entropy production (2) has the following algebraic structure:

$$\partial_\mu s^\mu \stackrel{!}{=} \sum_{i=1}^N \frac{\pi_{i,\alpha\beta} \pi_i^{\alpha\beta}}{2\eta_i T_i} \geq 0. \quad (3)$$

This leads to the dynamical equation for each  $\pi_i^{\alpha\beta}$ :

$$u^\mu \partial_\mu \pi_i^{\alpha\beta} = -\frac{\pi_i^{\alpha\beta}}{2\eta_i \beta_i} - \pi_i^{\alpha\beta} \frac{T_i}{\beta_i} \partial_\mu \left( \frac{\beta_i}{2T_i} u^\mu \right) + \frac{\sigma^{\alpha\beta}}{\beta_i}, \quad (4)$$

which was introduced by Israel and Stewart for a one-component system ( $N = 1$ ) [4, 5]. The main subject of this Letter is to confirm the validity of Eq. (4) for a multi-component system ( $N > 1$ ). Before we do so, we shall first determine the coefficients  $\eta_i$ , which in general differ from the usual definition of the shear viscosity. The reason for this is that  $\pi_i^{\alpha\beta}$ 's are correlated due to interactions between particles from different species. These correlations between  $\pi_i^{\alpha\beta}$ 's can only be seen, when each  $\eta_i$  depends on all  $\pi_j^{\alpha\beta}$ ,  $j = 1, 2, \dots, N$ . We will also show later that the  $\eta_i$  become the shear viscosities, when the ratios of components of  $\pi_i^{\alpha\beta}$ 's are relaxing to constants in time.

We now make use of relativistic kinetic theory and express the entropy current via the particle distribution function in phase space  $f_i(x, p_i)$ :

$$s^\mu = \sum_{i=1}^N \int d\Gamma_i p_i^\mu f_i(x, p_i) [1 - \ln f_i(x, p_i)] \quad (5)$$

with  $d\Gamma_i = d^3p_i/E_i/(2\pi)^3$ . It was shown for the case of  $N = 1$  [12] and is obviously true for  $N > 1$  that using the Grad's ansatz [5]  $f_i(x, p_i) = f_{i,eq}(x, p_i)(1 + A_i \pi_{i,\mu\nu} p_i^\mu p_i^\nu)$  in Eq. (5) one obtains Eq. (1) up to second order in  $\pi_{i,\mu\nu}$ . Here  $f_{i,eq}(x, p_i)$  is the equilibrium distribution function and  $A_i = [2(e_i + P_i)T_i^2]^{-1}$ , where  $P_i$  is the pressure.

Assuming that space-time evolutions of  $f_i(x, p_i)$  obey the Boltzmann equations

$$p_i^\mu \partial_\mu f_i = C_i[f_1, f_2, \dots, f_N] = C_{ii}[f_i] + \sum_{j=1, j \neq i}^N C_{ij}[f_i, f_j], \quad (6)$$

where  $C_{ii}$  are the collision terms describing interactions of particles of same species and  $C_{ij}$  describing binary interactions of particles of different species. Explicit expressions for the collision terms can be found for example in [13]. Taking derivative of (5) and using (6) we obtain

$$\partial_\mu s^\mu = \sum_{i=1}^N A_i \pi_{i,\mu\nu} \int d\Gamma_i p_i^\mu p_i^\nu C_i. \quad (7)$$

Comparison between Eqs. (7) and (3) leads to

$$\eta_i = \frac{\pi_{i,\mu\nu} \pi_i^{\mu\nu}}{2A_i \pi_{i,\mu\nu} \int d\Gamma_i p_i^\mu p_i^\nu C_i}. \quad (8)$$

Because the collision term  $C_i$  is a functional of all  $f_j$ 's, each  $\eta_i$  depends on all  $\pi_j^{\mu\nu}$ 's with  $j = 1, 2, \dots, N$ .

Equations (4) together with (8) are the main findings in this work. They provide the dynamic evolution of the shear pressure for each component of the mixture. This

is highly relevant for the present research on high energy nucleus-nucleus collisions at RHIC and LHC, where the produced quark-gluon plasma is a mixture, and thermal equilibration of quarks and gluons is expected to proceed differently during the expansion. For a one-component system, the shear viscosity  $\eta$ , or practically the shear viscosity to the entropy density ratio  $\eta/s$ , can be regarded as a to be specified but free external parameter. In contrast to that, for a multi-component system we recognize that each transport coefficient  $\eta_i$  depends on shear pressures of all components and, thus, is not a free parameter. This is a new property, which is not observed for a one-component system, although the basis of the derivation of Eqs. (4) is the same as in the second-order Israel-Stewart theory for a one-component system, namely the entropy principle Eq. (3). One may ask the question whether it is still correct to regard a multi-component mixture as an effective one-component system with a freely chosen  $\eta/s$ , as mostly done in Refs. [9] for the quark-gluon plasma.

In the rest of this Letter we will discuss some applications of the obtained equations. To confirm our findings, we solve the multi-component hydrodynamic equations and compare the results with those from kinetic transport theory.

For reasons of simplification we consider a one-dimensional expansion with the boost-invariance. In this case  $u^\mu = (t, 0, 0, z)/\tau$ , where  $\tau = \sqrt{t^2 - z^2}$  is the proper time. Because of the transversal isotropy the shear stress tensor is then  $\pi_i^{\mu\nu} = \text{diag}(0, \pi_i/2, \pi_i/2, -\pi_i)$  in the co-moving frame.  $\pi_i$  denotes the shear pressure of species  $i$ . The hydrodynamic equations (4) are reduced to

$$\dot{\pi}_i = -\frac{2}{9} \frac{\pi_i e_i}{\eta_i} - \frac{4}{3} \frac{\pi_i}{\tau} + \frac{8}{27} \frac{e_i}{\tau}. \quad (9)$$

$\dot{\pi}_i$  is the derivative with respect to  $\tau$ . The set of equations is the multi-component analog of the known Israel-Stewart second-order equations [4, 5, 12, 14] and has a relaxation-type form, existence of which was advocated in Ref. [15].

Furthermore, we consider only binary elastic collisions, which keep the particle number of each species conserved. Particles are assumed to be massless Boltzmann particles. We also take an isotropic distribution for the collision angle. With these simplifications we can analytically calculate the integrals in Eq. (8) and obtain (for details see [19])

$$\eta_i^{-1} = T_i^{-1} \sum_{j=1}^N \left( \frac{7}{6} \frac{n_j}{n_i} - \frac{1}{3} \frac{\pi_j}{\pi_i} \right) \sigma_{ij}, \quad (10)$$

where  $n_i$ 's are the local particle number densities,  $\sigma_{ij}$  denotes the cross section for a collision between a particle of species  $i$  and a particle of species  $j$ , and  $\sigma_{ii}$  is the cross section for a collision of two identical particles of species  $i$ . One can clearly see that  $\eta_i$  explicitly depends not only on temperature, cross sections, and the chemical

composition, but also on the ratios of the shear pressures  $\pi_j/\pi_i$ . Even when  $\pi_i$ 's become small compared with  $e_i$ 's, the ratios  $\pi_j/\pi_i$  cannot vanish.  $\eta_i$  is a coefficient rather than shear viscosity of the species  $i$  in the usual sense. We now want to address the question whether and how dissipation in a multi-component fluid as a whole can be described by a single effective shear viscosity coefficient.

Summing up the equations (9) over  $i$ , we obtain the hydrodynamic equation for the total shear pressure  $\pi = \sum_{i=1}^N \pi_i$ :

$$\dot{\pi} = -\frac{2}{9} \frac{\pi e}{\eta_{eff}} - \frac{4}{3} \frac{\pi}{\tau} + \frac{8}{27} \frac{e}{\tau} \quad (11)$$

with the definition of the effective shear viscosity for a mixture via

$$\frac{\pi e}{\eta_{eff}} = \sum_{i=1}^N \frac{\pi_i e_i}{\eta_i} = \frac{7}{2} \sum_{i,j=1}^N \pi_i n_j \sigma_{ij} - \sum_{i,j=1}^N \pi_j n_i \sigma_{ij}. \quad (12)$$

For the second identity the relation  $T_i = e_i/(3n_i)$  and Eq. (10) were used.  $e$  is the total local energy density. We recognize the equality of the two sums in (12) and thus can write

$$\eta_{eff} = \frac{2}{5} e \left( \sum_{i=1}^N \alpha_i \lambda_{mfp,i}^{-1} \right)^{-1}, \quad (13)$$

where  $\lambda_{mfp,i} = 1/\sum_{j=1}^N n_j \sigma_{ij}$  is the mean free path of particles of species  $i$  and  $\alpha_i = \pi_i/\pi$  is the fraction of the total shear pressure. For equal mean free paths,  $\lambda_{mfp,i} = \lambda$ , a multi-component system as a whole behaves as if all particles were identical. For this special case one finds  $\eta_{eff} = (2/5)(e/\lambda)$ , which is, indeed, exactly the result for a one-component system [14, 16], as it should be. For unequal mean free paths the effective shear viscosity additionally depends on the dynamic values  $\alpha_i$ . This means that in general, it is not possible to describe a mixture by a free parameter  $\eta_{eff}/s$ . One has to solve the equations (4) for each shear pressure.  $\eta/s$  is no longer a characteristic value. Mean free paths and temperatures of all particle species are the relevant scales.

On the other hand, the shear viscosity  $\eta$  of a mixture can be calculated by means of the Green-Kubo formula [17] and no dependence of  $\eta$  on the ratios  $\alpha_i$ 's is expected. However, the Green-Kubo formula is applicable when the particle system is in thermal equilibrium. Equation (13) gives a more general expression for the case the system is out of equilibrium. When the system is approaching equilibrium,  $\alpha_i$ 's become constants, and their values can be obtained by solving  $\dot{\alpha}_i = 0$  using Eqs. (9). The calculation is difficult, because we are dealing with a system of non-linear equations. We can solve it for the simple case of  $N = 2$  and obtain:

$$\frac{\pi_1}{\pi_2} \bigg|_{\alpha_1=\alpha_2=0} = \sqrt{\gamma^2 + \frac{n_1}{n_2}} - \gamma, \quad (14)$$

where

$$\gamma = \frac{5}{4n_2\sigma_{12}} \left( \frac{1}{\lambda_{mfp,1}} - \frac{1}{\lambda_{mfp,2}} \right) + \frac{1}{2} \left( 1 - \frac{n_1}{n_2} \right). \quad (15)$$

Putting (14) into (13) gives the shear viscosity of a two-component mixture in equilibrium,  $\eta_{eq}$ . We would like to mention that the weights  $\alpha_i$  in Eq. (13) depend not only on the composition  $n_1/n_2$ , but also on the difference between the inverses of the mean free paths, which is not trivial.

To verify our findings we now compare the solutions of hydrodynamic equations with those from kinetic transport calculations using the partonic cascade model BAMPS [13]. Previously published works demonstrated that BAMPS results can be regarded as a benchmark for hydrodynamic calculations [12, 18].

As a further simplification we consider a spatially homogeneous two-component particle system. This means that terms containing the gradient  $\partial_\mu u^\mu = 1/\tau$  in Eqs. (9) and (11) are dropped. Numerically this is realized by confining particles in a static box.

Initial condition is chosen as follows:  $T_1 = T_2 = 400$  MeV,  $n_1/n_2 = 5$ ,  $\pi_1/e_1 = \pi_2/e_2 = 0.3$ . Particles of species 1 are in chemical equilibrium, whereas particles of species 2 are undersaturated. Cross sections are  $\sigma_{11} = 3.88$  mb,  $\sigma_{22} = \sigma_{11}/4$ , and  $\sigma_{12} = \sigma_{11}/2$ . We thus mimic elastic interactions among gluons (species 1) and quarks (species 2).

Figure 1(a) shows the time evolutions of the partial shear pressures  $\pi_1$  and  $\pi_2$ , obtained by solving the hydrodynamic equations (9) (lines) and from BAMPS (symbols). Results are divided by total energy density. We see perfect agreement between the two approaches over up to 4 orders of magnitude. This demonstrates the validity of the new multi-component hydrodynamic equations derived here. To achieve such high numerical accuracy in BAMPS calculations we set the box to have a size of  $16 \times 16 \times 16$  fm<sup>3</sup> and use a test particle number of  $2 \cdot 10^2$ . The total particle number in one run is  $1.3 \cdot 10^7$ , and the results shown are the average obtained from  $10^4$  runs.

Results for the total shear pressure divided by total energy density are also shown in Fig. 1(a), compared to solution of Eq. (11) with the shear viscosity in thermal equilibrium  $\eta_{eq}$ . We see that the effective one-component hydrodynamic description underestimates the thermal equilibration at early times, while it can be applied at late times when the system is approaching equilibrium. In order to meet the BAMPS results at both initial and final times, one has to choose a smaller  $\eta_{eq}$ . This gives rise to conclusion that the extraction of the  $\eta/s$  value for the quark-gluon plasma at RHIC should be reexamined.

Finally, the  $\pi_1/\pi_2$  ratio is depicted in Fig. 1(b). Starting from the initial  $\pi_1/\pi_2 = 5$ , the ratio indeed relaxes to the value calculated analytically,  $\pi_1/\pi_2 = 0.488$ . [see Eq. (14)]. The effective relaxation time for  $\pi_1/\pi_2$  in this case is approximately 0.6 fm/c and has a complicated

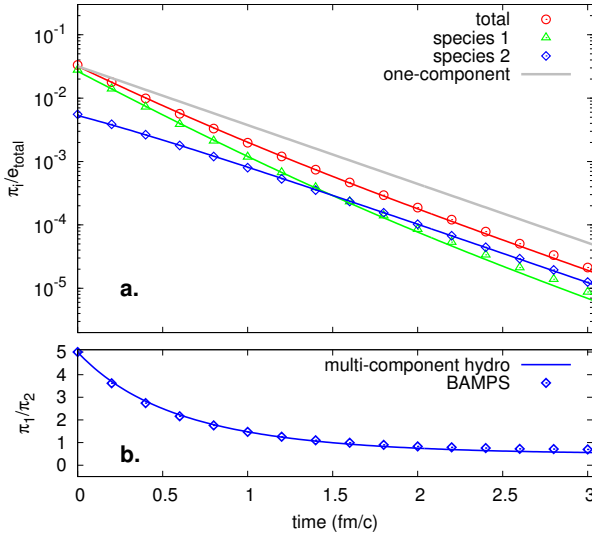


FIG. 1. (Color online) (a) Time evolution of the partial and total shear pressures normalized by total energy density and (b) ratio of the partial shear pressures from BAMPS (symbols) and hydrodynamic calculations (lines). Solid grey lines represent the effective one-component solution (see text).

dependence on densities and cross sections.

In this Letter we have derived novel hydrodynamic equations for each component of a multi-component system. Although the equations possess the relaxation form similar to that for a one-component system, they do reveal a new property – the shear viscosity coefficients for each component as well as the effective shear viscosity of the mixture depend on the ratios of shear pressures. This leads to the conclusion that hydrodynamic behaviour of a multi-component system as a whole in general cannot be described by one-component hydrodynamic equations with an effective shear viscosity coefficient calculated using standard methods (e.g. Green-Kubo relations). It is inevitable to solve the hydrodynamic equations for each component. Instead of  $\eta/s$ , temperatures and mean free paths are the relevant scales. We have confirmed our findings by comparing the solutions of the derived hydrodynamic equations with those from kinetic transport calculations using BAMPS. Both results agree with each other with high accuracy. Although the confirmation has been demonstrated for a static medium with simplifications on cross sections, our findings are expected to hold for general cases. A great potential of the present work lies in investigations of hydrodynamic behaviour of the quark gluon plasma at RHIC and LHC [19].

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- [1] S. S. Adler *et al.* [ PHENIX Collaboration ], Phys. Rev. Lett. **91**, 182301 (2003). [nucl-ex/0305013].
- [2] S. A. Voloshin, A. M. Poskanzer and R. Snellings, arXiv:0809.2949 [nucl-ex].
- [3] K. Aamodt *et al.* [The ALICE Collaboration], arXiv:1011.3914 [nucl-ex].
- [4] W. Israel, Ann. Phys. (N.Y.) **100** (1976) 310; J.M. Stewart, Proc. R. Soc. London, Ser. A **357** (1977) 59; W. Israel, M. Stewart, Ann. Phys. (N.Y.) **118** (1979) 341.
- [5] A. Muronga, Phys. Rev. **C69**, 034903 (2004). [arXiv:0309055 [nucl-th]].
- [6] P. Romatschke, Int. J. Mod. Phys. **E19**, 1-53 (2010). [arXiv:0902.3663 [hep-ph]].
- [7] G. S. Denicol, T. Koide, D. H. Rischke, Phys. Rev. Lett. **105**, 162501 (2010). [arXiv:1004.5013 [nucl-th]].
- [8] A. Monnai, T. Hirano, Nucl. Phys. **A847**, 283-314 (2010). [arXiv:1003.3087 [nucl-th]].
- [9] H. Song and U. W. Heinz, Phys. Rev. **C77**, 064901 (2008) [arXiv:0712.3715 [nucl-th]]; M. Luzum and P. Romatschke, Phys. Rev. C **78**, 034915 (2008) [Erratum-ibid. C **79**, 039903 (2009)] [arXiv:0804.4015 [nucl-th]]; D. A. Teaney, [arXiv:0905.2433 [nucl-th]]; H. Song, S. A. Bass and U. W. Heinz, arXiv:1103.2380 [nucl-th].
- [10] Z. Xu, C. Greiner and H. Stöcker, Phys. Rev. Lett. **101**, 082302 (2008) [arXiv:0711.0961 [nucl-th]].
- [11] Kranyš, M. 1970, Archive for Rational Mechanics and Analysis, **39**, 245 .
- [12] A. El, Z. Xu and C. Greiner, Phys. Rev. C **81**, 041901 (2010) [arXiv:0907.4500 [hep-ph]].
- [13] Z. Xu, C. Greiner, Phys. Rev. **C71**, 064901 (2005). [arXiv:0406278 [hep-ph]]; Phys. Rev. C **76**, 024911 (2007) [arXiv:hep-ph/0703233].
- [14] P. Huovinen and D. Molnar, Phys. Rev. C **79**, 014906 (2009) [arXiv:0808.0953 [nucl-th]].
- [15] G. S. Denicol, J. Noronha, H. Niemi and D. H. Rischke, [arXiv:1102.4780 [hep-th]].
- [16] S. R. de Groot, W. A. van Leeuwen, Ch. G. van Weert, Relativistic Kinetic Theory: Principles and Applications, North Holland, Amsterdam, 1980.
- [17] N. Demir and S. A. Bass, Phys. Rev. Lett. **102**, 172302 (2009) [arXiv:0812.2422 [nucl-th]]; J. I. Fuini, N. S. Demir, D. K. Srivastava and S. A. Bass, J. Phys. G **38**, 015004 (2011) [arXiv:1008.2306 [nucl-th]].
- [18] I. Bouras *et al.*, Phys. Rev. Lett. **103**, 032301 (2009) [arXiv:0902.1927 [hep-ph]]; Phys. Rev. **C82**, 024910 (2010). [arXiv:1006.0387 [hep-ph]].
- [19] A. El et. al., in preparation